J. Math. Anal. Appl. 376 (2011) 675–685

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On Γ-convergence of pairs of dual functionals ☆

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ARTICLE INFO

Article history:
Received 30 March 2010
Available online 14 October 2010
Submitted by A. Dontchev

Keywords:
Γ-convergence
Convex functionals
Duality

ABSTRACT

The paper considers a slightly modified notion of the Γ-convergence of convex functionals in uniformly convex Banach spaces and establishes that under standard coercitivity and growth conditions the Γ-convergence of a sequence of functionals $\{F_j\}$ to $\tilde{F}$ implies that the corresponding sequence of dual functionals $\{F_j^*\}$ converges in an analogous sense to the dual to $\tilde{F}$ functional $\tilde{F}^*$.

1. Introduction

In the convex analysis it is well known that the duality relationship defined by the Young–Fenchel transform $F \to F^*$ in a reflexive Banach space is preserved under the Mosco convergence.

More precisely, let X be a reflexive Banach space, X^* be its dual space, $\langle\cdot,\cdot\rangle$ be the pairing between X and X^* and let $\mathcal{G}(X)$ ($\mathcal{G}(X^*)$) be the set of all proper lower semicontinuous convex functions on X (resp. X^*). The Young–Fenchel transform $F \to F^*$ from $\mathcal{G}(X)$ to $\mathcal{G}(X^*)$ is defined as

$$F^*(x^*) := \sup_{x\in X}\bigl[\langle x^*, x\rangle - F(x)\bigr], \quad x^* \in X^*.$$

A sequence $\{F_k\} \subset \mathcal{G}(X)$ is said to be Mosco convergent to $F_0 \in \mathcal{G}(X)$ if for every $x \in X$ the following properties hold:

(i) $F_0(x) \leqslant \lim_{k\to\infty} \inf F_k(x_k)$ whenever $x_k \rightharpoonup x$ weakly in X;
(ii) there exists a sequence (recovery sequence) $\{x_k\} \subset X$, which converges strongly to x, such that $F_0(x) = \lim_{k\to\infty} F_k(x_k)$.

Exactly in the same way the Mosco convergence on $\mathcal{G}(X^*)$ is defined. We shall denote these convergences by $\lim(M)F_k = F$ or $\lim(M)\mathcal{F}_k = \mathcal{F}$ for $\{F_k\} \subset \mathcal{G}(X)$ and $\{\mathcal{F}_k\} \subset \mathcal{G}(X^*)$ respectively. In these notations the preservation of the duality relationship means that

$$\Bigl(\lim_{k\to\infty}(M)F_k\Bigr)^* = \lim_{k\to\infty}(M)F_k^*$$

whenever $\{F_k\} \subset \mathcal{G}(X)$ Mosco converges to some $F_0 \in \mathcal{G}(X)$. This "sequential continuity" for reflexive Banach spaces was established in Mosco [13]. After that, a series of papers followed, where this relationship was extended to more general cases and topologies were described with respect to which the Young–Fenchel transform is continuous, see, for instance, Beer [2].

☆ This work was supported by the Latvian Council of Sciences under grant 09.1579 and by the ESF Project No. 2009/0223/1DP/1.1.1.2.0/APIA/VIAA/008.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.10.033

Zhikov [17] performed a different approach, partially caused by the fact that the weaker notion of Γ-convergence, see, for instance, Dal Maso [6], gives better compactness properties. In [17] it was shown that the introduced there $\Gamma(V)$-convergence of convex integral functionals of the form

$$I(u) := \int_{\Omega} f\big(x, \nabla u(x)\big)\, dx$$

preserves duality properties between densities of energy and densities of complementary energies even for some classes of integrands with nonstandard growth conditions

$$\nu|\xi|^q - c \leqslant f(x,\xi) \leqslant |\xi|^p + c \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbf{R}^n,$$

with $1 < q \leqslant p$.

The approach developed by Zhikov [17], namely, the one considering sequences of functionals $F_k : X \to \mathbf{R}$, $k = 1, 2, \dots$, with respect to the weak convergence of elements from some subspace $V \subset X$, i.e. limits of the kind

$$F(x) := \lim_{k\to\infty} \inf F_k(x + v_k), \quad v_k \rightharpoonup 0 \text{ weakly in } V,$$

seems very suitable for many physical and mechanical problems, where the principle of the minimal energy is formulated in terms of convex integral functionals. The framework of spaces $X := L_p(\Omega; \mathbf{R}^n)$ and $V := \{v \in X \mid v = \nabla u,\ u \in W_p^1(\Omega)\}$ is natural for these problems and various methods of *a posteriori* error estimates for numerical schemes are developed within this framework, see, for instance, Carstensen [5], Han [11] and Repin [16]. Besides, the knowledge of the preservation of duality relationships between the primal and the complementary energies under $\Gamma(V)$-convergence and the duality between curl-free and div-free function spaces would help to establish properties of convergence of functionals on div-free vector spaces via known properties of convergence of functionals on curl-free vector spaces.

In this paper, we show that the preservation of duality properties holds for some classes of convex functionals on uniformly convex separable Banach spaces with respect to suitable $\Gamma(V)$-convergences with better compactness properties than the Mosco convergence.

To do that, we adapt the approach by Zhikov [17] and consider a separable uniformly convex Banach space X and its closed linear subspace V, by means of which we introduce the following notion of $\Gamma(V)$-convergence.

Definition 1.1. A sequence $\{F_j\}$ of functionals $F_j : X \to \mathbf{R}$, $j = 1, 2, \dots$, $\Gamma(V)$-converges to a functional $\tilde{F} : X \to \mathbf{R}$ iff

(i) for every $x \in X$ and every sequence $\{v_j\} \subset V$ that converges weakly in X to zero as $j \to \infty$ there is

$$\tilde{F}(x) \leqslant \liminf_{j\to\infty} F_j(x + v_j); \tag{1}$$

(ii) for every $x \in X$ there exists a sequence (a recovery sequence) $\{v_j\} \subset V$ that converges weakly in X to zero as $j \to \infty$ such that

$$\tilde{F}(x) = \lim_{j\to\infty} F_j(x + v_j). \tag{2}$$

Analogously, for the subspace $\mathcal{N} \subset X^*$,

$$\mathcal{N} := \big\{\eta \in X^* \bigm| \langle \eta, v\rangle = 0\ \forall v \in V\big\},$$

we define the $\Gamma(\mathcal{N})$-convergence of sequences of functionals $\mathcal{F}_j : X^* \to \mathbf{R}$, $j = 1, 2, \dots$.

Definition 1.2. A sequence $\{\mathcal{F}_j\}$ of functionals $\mathcal{F}_j : X^* \to \mathbf{R}$, $j = 1, 2, \dots$, $\Gamma(\mathcal{N})$-converges to a functional $\tilde{\mathcal{F}} : X^* \to \mathbf{R}$ iff

(i) for every $x^* \in X^*$ and every sequence $\{\eta_j\} \subset \mathcal{N}$ that converges weakly in X^* to zero as $j \to \infty$ there is

$$\tilde{\mathcal{F}}\big(x^*\big) \leqslant \liminf_{j\to\infty} \mathcal{F}_j\big(x^* + \eta_j\big);$$

(ii) for every $x^* \in X^*$ there exists a sequence (a recovery sequence) $\{\eta_j\} \subset \mathcal{N}$ that converges weakly to zero in X^* as $j \to \infty$ such that

$$\tilde{\mathcal{F}}\big(x^*\big) = \lim_{j\to\infty} \mathcal{F}_j\big(x^* + \eta_j\big).$$

Here and in what follows, by $\langle\cdot,\cdot\rangle$ we denote the pairing between X and its dual space X^*, and for a functional $F : X \to \mathbf{R}$ by F^* we will denote the dual (conjugate) functional

$$F^*\big(x^*\big) := \sup_{x\in X}\big(\big\langle x^*, x\big\rangle - F(x)\big), \quad x^* \in X^*.$$

We recall that a Banach space is said to be uniformly convex if

$$\omega(t) := \inf\bigl\{1 - \bigl\|1/2(x+y)\bigr\| \bigm| \|x\| = 1 = \|y\|,\ \|x-y\| \geqslant t\bigr\} > 0 \quad \forall t > 0.$$

Obviously, the function ω is non-decreasing for $t > 0$. We will use this property later. Since uniformly convex Banach spaces are reflexive, see, for instance, Milman [12], the space X^* also is separable and the subspaces V and $\mathcal{N}$ are mutually "orthogonal", i.e.

$$V = \bigl\{v \in X \bigm| \langle \eta, v\rangle = 0\ \forall \eta \in \mathcal{N}\bigr\},$$

see, for instance, [8].

To describe classes of eligible functionals, we will use the following definition.

Definition 1.3. For non-negative strictly convex and continuously differentiable functions $\gamma_i : \mathbf{R} \to \mathbf{R}$ with $\gamma_i(0) = 0$ and $\gamma_i(t)/|t| \to \infty$ as $|t| \to \infty$, and constants c_i, $i = 1, 2$, the class $\mathcal{M}(\gamma_1, \gamma_2, c_1, c_2)$ consists of all convex functionals $F : X \to \mathbf{R}$ such that

$$\gamma_1\bigl(\|x\|\bigr) - c_1 \leqslant F(x) \leqslant \gamma_2\bigl(\|x\|\bigr) + c_2, \quad x \in X. \tag{3}$$

The properties of functions γ_i, $i = 1, 2$, ensure that their conjugate functions γ_i^*,

$$\gamma_i^*(\tau) := \sup\bigl\{\tau t - \gamma_i(t) \bigm| t \in \mathbf{R}\bigr\}, \quad \tau \in \mathbf{R},\ i = 1, 2,$$

have the same properties as γ_i. Further, simple standard calculations give that from $F \in \mathcal{M}(\gamma_1, \gamma_2, c_1, c_2)$ it follows that its dual functional F^* belongs to the class $\mathcal{M}^*(\gamma_1, \gamma_2, c_1, c_2)$ of convex functionals $\mathcal{F} : X^* \to \mathbf{R}$ such that

$$\gamma_2^*\bigl(\|x^*\|\bigr) - c_2 \leqslant \mathcal{F}\bigl(x^*\bigr) \leqslant \gamma_1^*\bigl(\|x^*\|\bigr) + c_1, \quad x^* \in X^*. \tag{4}$$

In this section and the next one we suppose that the functions γ_1, γ_2 and the constants c_1, c_2 are fixed, and, for the sake of simplicity of abbreviations only, we will omit references to them and will write simply $\mathcal{M}$ or $\mathcal{M}^*$ instead of $\mathcal{M}(\gamma_1, \gamma_2, c_1, c_2)$ or $\mathcal{M}^*(\gamma_1, \gamma_2, c_1, c_2)$ respectively. For standard notions and properties of convex functionals we refer to [9] or [10].

The main results of the paper are the following theorems.

Theorem 1.4. *Let X be a separable uniformly convex Banach space. Let $\{F_k\} \subset \mathcal{M}$ be a countable sequence. Then there exist a subsequence $\{F_j\} \subset \{F_k\}$ and a functional $\tilde{F} \in \mathcal{M}$ such that the sequence $\{F_j\}$ $\Gamma(V)$-converges to $\tilde{F}$.*

Theorem 1.5. *Let X be a separable uniformly convex Banach space and let a sequence $\{F_j\} \subset \mathcal{M}$ $\Gamma(V)$-converge to a functional $\tilde{F}$. Then the sequence of corresponding dual functionals $\{F_j^*\}$ $\Gamma(\mathcal{N})$-converges to the dual to $\tilde{F}$ functional $(\tilde{F})^* \in \mathcal{M}^*$.*

The proofs of these theorems are presented in Section 2, and in Section 3 we briefly discuss two classes of integral functionals of the calculus of variations to which the general results apply. For the convenience of the reader we give a short sketch of proofs for the integral representation of the $\Gamma(V)$-limit functionals for integral functionals of the second class of the type

$$u \to \int_\Omega f\bigl(x, \nabla u(x)\bigr)\, dx + \int_{\partial\Omega} g\bigl(x, u(x)\bigr)\, dS, \quad u \in W_2^1(\Omega) \cap L_p(\partial\Omega).$$

Finally, we present a simple sequence of nonlinear coercive integral functionals, which $\Gamma(V)$-converges but does not contain Mosco convergent subsequences.

2. General case

In this section we present proofs of Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Let $\{F_k\} \subset \mathcal{M}$ be a countable sequence. By virtue of the coercitivity condition (3), we can assume, without loss of generality, that for every fixed $x \in X$ the sequences $\{v_j\}$ in (1) and (2) belong to a bounded set. Due to the reflexivity and the separability of X, in every bounded set $\{x \in X \mid \|x\| \leqslant c\}$ the weak convergence can be defined by a metric. Hence, the standard diagonal process gives that there exist a subsequence $\{F_j\} \subset \{F_k\}$ and a countable set $\{x_l\}$ that is dense in X such that for every x_l, $l = 1, 2, \ldots,$ there exists a (recovery) sequence $\{v_{lj}\} \subset V$, $v_{lj} \rightharpoonup 0$ weakly as $j \to \infty$, such that

$$\tilde{F}(x_l) := \inf\Bigl\{\liminf_{j\to\infty} F_j(x_l + v_j) \Bigm| \{v_j\} \subset V,\ v_j \rightharpoonup 0 \text{ weakly as } j \to \infty\Bigr\} = \lim_{j\to\infty} F(x_l + v_{lj}).$$

Now we define the functional $\tilde F : X \to \mathbf{R}$ as

$$\tilde F(x) := \inf\Big\{\liminf_{j\to\infty} F_j(x+v_j) \;\Big|\; \{v_j\}\subset V,\ v_j \rightharpoonup 0 \text{ weakly as } j\to\infty\Big\}. \tag{5}$$

Properties of functionals from $\mathcal{M}$ ensure that the functional $\tilde F$ is well defined and that $\tilde F$ satisfies estimates (3).

Indeed, by inserting $v_j = 0$, $j = 1, 2, \ldots,$ in (5) we obtain the right-hand side estimate in (3) for $\tilde F$. In its turn, the weak lower semicontinuity of the mapping $x \to \gamma_1(\|x\|)$ provides the left-hand side estimate in (3) for $\tilde F$. Moreover, from convexity of F_j, $j = 1, 2, \ldots,$ and (5) it follows that $\tilde F$ is a convex functional, too. To prove that, let us fix two elements $x', x'' \in X$. For every $\varepsilon > 0$ from (5) follows the existence of eligible sequences $\{v'_j\}$ and $\{v''_j\}$ such that

$$\tilde F(x') \geqslant \liminf_{j\to\infty} F_j(x'+v'_j) - \varepsilon,$$
$$\tilde F(x'') \geqslant \liminf_{j\to\infty} F_j(x''+v''_j) - \varepsilon.$$

From here, the convexity of involved functionals and (5) it follows

$$\begin{aligned}\tilde F\big(1/2(x'+x'')\big) &\leqslant \liminf_{j\to\infty} F_j\big(1/2(x'+x'') + 1/2(v'_j+v''_j)\big)\\ &\leqslant 1/2\tilde F(x') + 1/2\tilde F(x'') + \varepsilon,\end{aligned}$$

and arbitrarity of $\varepsilon > 0$ gives the desired estimate for $\tilde F$. Thus $\tilde F \in \mathcal{M}$.

The convexity of $\tilde F$ and the growth estimate (3) ensure that $\tilde F$ is locally Lipschitz. More than that, the growth estimates for $F \in \mathcal{M}$ give, see, for instance, [9], that for every $R > 0$ there exists a constant L_R, which depends only on the parameters of the class $\mathcal{M}$ and R, such that

$$\big|F(x) - F(y)\big| \leqslant L_R\|x-y\| \quad \text{whenever } \|x\| \leqslant R,\ \|y\| \leqslant R.$$

If x_0 is an element of X, then there exists a subsequence $\{x_i\} \subset \{x_l\}$ that converges strongly to x_0, and, by continuity of $\tilde F$, the standard diagonal process gives

$$\tilde F(x_0) = \lim_{i\to\infty} \tilde F(x_i) = \lim_{i\to\infty}\lim_{j\to\infty} F_j(x_i + v_{ij}) = \lim_{j\to\infty} F_j(x_0 + v_{0j})$$

for some sequence $\{v_{0j}\} \subset V$, which converges to zero weakly as $j \to \infty$. This property together with the definition of $\tilde F$ by (5) gives that the sequence $\{F_j\}$ $\Gamma(V)$-converges to $\tilde F$. This completes the proof of Theorem 1.4. □

Before proceeding with the proof of Theorem 1.5, we establish two preliminary lemmas.

Lemma 2.1. *Let X be a separable uniformly convex Banach space and let $\varepsilon > 0$ be fixed. If the sequence $\{G_j\}$, where*

$$G_j(x) := F_j(x) + \varepsilon\gamma_1\big(\|x\|\big), \quad x \in X,\ F_j \in \mathcal{M},\ j = 1, 2, \ldots,$$

$\Gamma(V)$-converges to a functional $\tilde G$, then the functional $\tilde G$ is strictly convex, i.e.

$$\tilde G\big(1/2(x+y)\big) < 1/2\tilde G(x) + 1/2\tilde G(y) \quad \textit{if } x \neq y.$$

Proof. Let us suppose the contrary, i.e. that there exist elements $x_1, x_2 \in X$, $x_1 \neq x_2$, such that

$$\tilde G\big(1/2(x_1+x_2)\big) = 1/2\tilde G(x_1) + 1/2\tilde G(x_2).$$

Without loss of generality we can assume that

$$\tilde G(x) \geqslant 0 \quad \forall x \in X, \qquad \tilde G(x_1) = \tilde G\big(1/2(x_1+x_2)\big) = \tilde G(x_2) = 0. \tag{6}$$

Otherwise we can use the transform $G \Longrightarrow \hat G$,

$$\hat G(x) := G(x) - \big\langle l^*, x - 1/2(x_1+x_2)\big\rangle - \tilde G\big(1/2(x_1+x_2)\big),$$

where $l^* \in \partial\tilde G(1/2(x_1+x_2))$. By $\partial F(x)$ here and in the sequel we denote the subdifferential of the functional F at the element x.

Let $\{v_{1j}\}$ and $\{v_{2j}\}$ be recovery sequences for $\tilde G(x_1)$ and $\tilde G(x_2)$ respectively. From (6) it follows that the sequence $\{1/2(v_{1j}+v_{2j})\}$ is the recovery sequence for $\tilde G(1/2(x_1+x_2))$. Indeed, if it is not so, then there exist a subsequence of indices $\{j'\} \subset \{j\}$ and a positive constant $d > 0$ such that

$$
\begin{aligned}
0 = \tilde{G}\big(1/2(x_1+x_2)\big) &\leqslant \liminf_{j'\to\infty} G_{j'}\big(1/2(x_1+x_2)+1/2(v_{1j'}+v_{2j'})\big) - d \\
&\leqslant \liminf_{j'\to\infty}\big[1/2G_{j'}(x_1+v_{1j'})+1/2G_{j'}(x_2+v_{2j'})\big] - d \\
&= 1/2\tilde{G}(x_1)+1/2\tilde{G}(x_2) - d,
\end{aligned}
$$

what contradicts with (6).

Let us denote

$$z_{1j} := x_1 + v_{1j}, \quad z_{2j} := x_2 + v_{2j}, \quad z_j := 1/2(z_{1j}+z_{2j}), \quad j = 1,2,\dots.$$

Without loss of generality we can assume that the sequences $\{\|z_{1j}\|\}$, $\{\|z_{2j}\|\}$ and $\{\|z_j\|\}$ converge to d_1, d_2 and d_0 respectively. Obviously, there exist sequences $\{y_{1j}\}$, $\{y_{2j}\}$, $\{y_j\} \subset X$, which converge to zero strongly as $j \to \infty$, such that

$$\|z_{1j}+y_{1j}\| = d_1, \quad \|z_{2j}+y_{2j}\| = d_2, \quad \|z_j+y_j\| = d_0, \quad j = 1,2,\dots.$$

From properties of recovery sequences and the convexity of functionals $F \in \mathcal{M}$ we have

$$
\begin{aligned}
&G_j(z_j) \leqslant 1/2G_j(z_{1j}) + 1/2G_j(z_{2j}) + \varepsilon I_j, \\
&I_j := \gamma_1\big(1/2\|z_{1j}+z_{2j}\|\big) - 1/2\gamma_1\big(\|z_{1j}\|\big) - 1/2\gamma_1\big(\|z_{2j}\|\big), \quad j = 1,2,\dots, \\
&\tilde{G}\big(1/2(x_1+x_2)\big) \leqslant 1/2\tilde{G}(x_1) + 1/2\tilde{G}(x_2) + \varepsilon \liminf_{j\to\infty} I_j.
\end{aligned} \tag{7}
$$

If $d_1 = d_2 = 0$, then, obviously, $x_1 = x_2 = 0$.

If $d_1 = d_2 = d > 0$, then, by properties of γ_1 and the uniform convexity of X,

$$
\begin{aligned}
\liminf_{j\to\infty} I_j &= \liminf_{j\to\infty}\big[\gamma_1\big(1/2\|z_{1j}+y_{1j}+z_{2j}+y_{2j}\|\big) - 1/2\gamma_1\big(\|z_{1j}+y_{1j}\|\big) - 1/2\gamma_1\big(\|z_{2j}+y_{2j}\|\big)\big] \\
&\leqslant \liminf_{j\to\infty}\big[\gamma_1\big(d - \omega\big(1/d\|z_{1j}+y_{1j}-z_{2j}-y_{2j}\|\big)\big) - 1/2\gamma_1(d) - 1/2\gamma_1(d)\big] \\
&\leqslant \gamma_1\big(d - \omega\big(1/d\|x_1-x_2\|\big)\big) - \gamma_1(d) < 0,
\end{aligned}
$$

what contradicts with (6) and (7).

If $d_2 = d_1 + d$, $d > 0$, then, analogously as above, we have

$$\liminf_{j\to\infty} I_j \leqslant \gamma_1(d_0) - 1/2\gamma_1(d_1) - 1/2\gamma_1(d_2) < 0,$$

because $d_0 \leqslant 1/2(d_1+d_2)$ and γ_1 is strictly convex and strictly increasing on $\mathbf{R}_+$. □

Lemma 2.2. *Let X be a uniformly convex Banach space. Let $F \in \mathcal{M}$. Then for every pair $(x_0, x_0^*) \in X \times X^*$ there exists a pair $(v_0, \eta_0) \in V \times \mathcal{N}$ such that*

$$F(x_0+v_0) - \langle x_0^*+\eta_0, x_0+v_0\rangle + F^*\big(x_0^*+\eta_0\big) = 0.$$

Proof. We recall that by the Fenchel's inequality

$$F(x) - \langle x^*, x\rangle + F^*\big(x^*\big) \geqslant 0 \quad \forall\big(x, x^*\big) \in X \times X^*,$$

and

$$F(x) - \langle x^*, x\rangle + F^*\big(x^*\big) = 0$$

iff $x \in \partial F^*(x^*)$ and $x^* \in \partial F(x)$. More than that, if $\partial F(x)$ is a singleton, i.e. $\partial F(x)$ consists of one element x^*, then F is Gateaux differentiable at x and the corresponding derivative $F'(x) = x^*$, see, for instance [9]. Obviously, analogous properties hold for $\mathcal{F} \in \mathcal{M}^*$.

Let the elements $(x, x^*) \in X \times X^*$ be fixed. We define for a fixed ε, $0 < \varepsilon < 1$, the functional $F_\varepsilon : X \to \mathbf{R}$,

$$F_\varepsilon(x) := F(x) + \varepsilon\gamma_1\big(\|x\|\big), \quad x \in X,$$

and let F_ε^* be the dual functional. From properties of functionals from classes $\mathcal{M}$ and $\mathcal{M}^*$ we have that F_ε and F_ε^* belong to analogous classes and that for every fixed $R > 0$

$$\sup_{\|x\|\leqslant R,\ \|x^*\|\leqslant R}\big[\big|F_\varepsilon(x) - F(x)\big| + \big|F_\varepsilon^*\big(x^*\big) - F^*\big(x^*\big)\big|\big] \to 0 \quad \text{as } \varepsilon \to 0. \tag{8}$$

By construction, the functional F_ε is strictly convex, hence, the subdifferentials $\partial F_\varepsilon^*(\cdot)$ are singletons and F_ε^* is Gateaux differentiable on X^*. Since the mapping

$$\eta \to F_\varepsilon^*\big(x_0^*+\eta\big) - \langle x_0^*+\eta, x_0\rangle, \quad \eta \in \mathcal{N},$$

is convex and coercive, and X^* is reflexive, then there exists $\eta_\varepsilon \in \mathcal{N}$ such that

$$F_\varepsilon^*\big(x_0^* + \eta_\varepsilon\big) - \big\langle x_0^* + \eta_\varepsilon, x_0\big\rangle \leqslant F_\varepsilon^*\big(x_0^* + \eta\big) - \big\langle x_0^* + \eta, x_0\big\rangle \quad \forall \eta \in \mathcal{N}.$$

From here, Gateaux differentiability of F_ε^* and "orthogonality" between V and $\mathcal{N}$ we have the existence of $v_\varepsilon \in V$ such that

$$F_\varepsilon^{*\prime}\big(x_0^* + \eta_\varepsilon\big) - x_0 = v_\varepsilon.$$

The above mentioned fact and the duality between F_ε and F_ε^* imply

$$F_\varepsilon\big(x_0^* + \eta_\varepsilon\big) - \big\langle x_0^* + \eta_\varepsilon, x_0 + v_\varepsilon\big\rangle + F_\varepsilon(x_0 + v_\varepsilon) = 0.$$

By equi-coercitivity of functionals F_ε and F_ε^* the set $\{(v_\varepsilon, \eta_\varepsilon) \in X \times X^* \mid 0 < \varepsilon < 1\}$ is bounded. Since X is reflexive, there exists a subsequence $\{v_{\varepsilon_k}, \eta_{\varepsilon_k}\}$ that converges weakly to a pair (v_0, η_0). From here, the convexity of F and F^* and (8) it follows immediately

$$F(x_0 + v_0) - \big\langle x_0^* + \eta_0, x_0 + v_0\big\rangle + F^*\big(x_0^* + \eta_0\big) = 0.$$

This completes the proof. □

Proof of Theorem 1.5. Let the sequence $\{F_j\} \subset \mathcal{M}$ $\Gamma(V)$-converge to $\tilde{F}$. Since the space X is reflexive and functionals from $\mathcal{M}^*$ possess analogous properties (the convexity and the growth estimates) as functionals from $\mathcal{M}$, there exists a subsequence $\{F_i^*\} \subset \{F_j^*\}$ that $\Gamma(\mathcal{N})$-converges to a functional $\mathcal{F} \in \mathcal{M}^*$. Without loss of generality we can assume that the whole sequence $\{F_j^*\}$ $\Gamma(\mathcal{N})$-converges to $\mathcal{F}$.

To establish that $\mathcal{F} = \tilde{F}^*$ it is sufficient to show that for every $\hat{x} \in X$ there exists $\hat{x}^* \in X^*$ such that

$$\tilde{F}(\hat{x}) - \big\langle \hat{x}^*, \hat{x}\big\rangle + \mathcal{F}\big(\hat{x}^*\big) \leqslant 0. \tag{9}$$

Indeed, from Definitions 1.1 and 1.2 of $\Gamma(V)$- and $\Gamma(\mathcal{N})$-convergence, the Fenchel's inequality and "orthogonality" between V and $\mathcal{N}$ we have

$$\tilde{F}(x) - \big\langle x^*, x\big\rangle + \mathcal{F}\big(x^*\big) \geqslant 0 \quad \forall\big(x, x^*\big) \in X \times X^*$$

and (9) implies that for every $\hat{x} \in X$

$$\tilde{F}(\hat{x}) = \sup_{x^* \in X^*}\big[\big\langle x^*, \hat{x}\big\rangle - \mathcal{F}\big(x^*\big)\big],$$

i.e. that $\mathcal{F}^* = \tilde{F}$, and, by the convexity of $\mathcal{F}$ and $\tilde{F}$, we have $\mathcal{F} = \tilde{F}^*$.

To proceed further with the proof, let us suppose that $\tilde{F}$ is a strictly convex functional. Then for every $x^* \in X^*$ there exists a unique element $\varphi(x^*) \in X$ such that

$$\tilde{F}\big(\varphi\big(x^*\big)\big) - \big\langle x^*, \varphi\big(x^*\big)\big\rangle \leqslant \tilde{F}(x) - \big\langle x^*, x\big\rangle \quad \forall x \in X.$$

Let us fix the element $x_0^* \in X^*$ and let $x_0 := \varphi(x_0^*)$. By virtue of Lemma 2.2, for every $j = 1, 2, \ldots,$ there exists a pair $(v_j, \eta_j) \in V \times \mathcal{N}$ such that

$$F_j(x_0 + v_j) - \big\langle x_0^* + \eta_j, x_0 + v_j\big\rangle + F_j^*\big(x_0^* + \eta_j\big) = 0, \quad j = 1, 2, \ldots. \tag{10}$$

We claim that the sequence $\{v_j\}$ converges weakly to zero as $j \to \infty$. Indeed, if V_0 is the set of cluster points for $\{v_j\}$ in the weak topology of X, then from Definition 1.1, the Fenchel's inequality and (10) it follows immediately

$$\begin{aligned}\tilde{F}(x_0) - \big\langle x_0^*, x_0\big\rangle &= \lim_{j\to\infty}\big[F_j(x_0 + v_{0j}) - \big\langle x_0^*, x_0 + v_{0j}\big\rangle\big] \\ &\geqslant \limsup_{j\to\infty}\big[F_j(x_0 + v_j) - \big\langle x_0^*, x_0 + v_j\big\rangle\big] \\ &\geqslant \sup_{v_0 \in V_0}\big[\tilde{F}(x_0 + v_0) - \big\langle x_0^*, x_0 + v_0\big\rangle\big],\end{aligned}$$

where $\{v_{0j}\}$ is the recovery sequence for $\tilde{F}(x_0)$. This inequality, strict convexity of $\tilde{F}$ and the choice of $x_0 = \varphi(x_0^*)$ imply that $V_0 = \{0\}$.

If η_0 is an arbitrary cluster point in the weak topology for the sequence $\{\eta_j\}$, then, after passing to the limit $j \to \infty$ in (10), we obtain

$$\tilde{F}(x_0) - \big\langle x_0^* + \eta_0, x_0\big\rangle + \mathcal{F}\big(x_0^* + \eta_0\big) \leqslant 0. \tag{11}$$

Therefore, for every $x_0^* \in X^*$ there exist elements $x_0 = \varphi(x_0^*) \in X$ and $\eta_0 \in X^*$ such that (11) holds.

If the range of the mapping $\varphi : X^* \to X$ is the whole X, then, for every chosen $\hat{x} \in X$, there exist $\hat{x}^* \in X^*$ with $\varphi(\hat{x}^*) = \hat{x}$ and, by (11), a corresponding element $\hat{\eta}$ such that

$$\tilde{F}(\hat{x}) - \big\langle \hat{x}^* + \hat{\eta}, \hat{x}\big\rangle + \mathcal{F}\big(\hat{x}^* + \hat{\eta}, \hat{x}\big) \leqslant 0,$$

i.e. (9) holds with $\hat{x}^* + \hat{\eta}$.

To show that the range of φ is the whole X we recall that the functional $\tilde{F}^*$ is coercive, hence, for every chosen $\tilde{x} \in X$ there is $\tilde{x}^* \in X^*$ such that

$$\tilde{F}^*(\tilde{x}^*) - \langle \tilde{x}^*, \tilde{x} \rangle \leqslant \tilde{F}^*(x^*) - \langle x^*, \tilde{x} \rangle \quad \forall x^* \in X^*.$$

By duality, $\tilde{F} = (\tilde{F}^*)^*$, i.e. the above inequality means that

$$\tilde{F}^*(\tilde{x}^*) - \langle \tilde{x}^*, \tilde{x} \rangle + \tilde{F}(\tilde{x}) = 0.$$

From here and the Fenchel's inequality

$$\tilde{F}(\tilde{x}) - \langle \tilde{x}^*, \tilde{x} \rangle \leqslant \tilde{F}(x) - \langle \tilde{x}^*, x \rangle \quad \forall x \in X,$$

i.e. $\tilde{x} = \varphi(\tilde{x}^*)$. Thus the range of φ is the whole X.

In the general case, i.e. it is not supposed that $\tilde{F}$ is a strictly convex functional, we proceed analogously as in the proof of Lemma 2.1.

Instead of the sequence $\{F_j\}$ we consider, for an arbitrary chosen $\varepsilon \in (0,1)$, a sequence $\{F_{\varepsilon j}\}$ with

$$F_{\varepsilon j}(x) := F_j(x) + \varepsilon \gamma_1(\|x\|), \quad x \in X, \ j = 1, 2, \ldots.$$

If it is necessary, after passing to a subsequence, we have that the sequence $\{F_{\varepsilon j}\}$ $\Gamma(V)$-converges to a functional $\tilde{F}_\varepsilon$, which now is strictly convex, and the corresponding sequence of dual functionals $\{F^*_{\varepsilon j}\}$ $\Gamma(\mathcal{N})$-converges to $(\tilde{F}_\varepsilon)^*$.

The same continuity argument as in the proof of Lemma 2.1 gives that for every $R > 0$ and $\delta > 0$ there exists $\varepsilon(R,\delta) > 0$ such that

$$\sup_{\|x\| \leqslant R, \, \|x^*\| \leqslant R} \left[\left| \tilde{F}_\varepsilon(x) - \tilde{F}(x) \right| + \left| (\tilde{F}_\varepsilon)^*(x^*) - \mathcal{F}(x^*) \right| \right] < \delta \quad \text{whenever } 0 < \varepsilon < \varepsilon(R,\delta).$$

From here follows an analogous estimate for the difference $|(\tilde{F}_\varepsilon)^*(x^*) - (\tilde{F})^*(x^*)|$ and, as a consequence, the estimate for the difference $|\mathcal{F}(x^*) - (\tilde{F})^*(x^*)|$, that, after passing to the limit $\varepsilon \to 0$, gives that $\mathcal{F} = (\tilde{F})^*$ in the general case.

To complete the proof of Theorem 1.5, we point out that due to the duality $F_j = (F_j^*)^*$, $j = 1, 2, \ldots,$ every $\Gamma(\mathcal{N})$-convergent subsequence $\{F_i^*\} \subset \{F_j^*\}$ converges to $(\tilde{F})^*$. From here it follows immediately that the whole sequence $\{F_j^*\}$ $\Gamma(\mathcal{N})$-converges to $(\tilde{F})^*$. □

Remark 2.3. It is easy to see that the above introduced notions of $\Gamma(V)$- and $\Gamma(\mathcal{N})$-convergence possess the well-known property of Γ-convergence of preserving the convergence of minimizers. Indeed, let a sequence $\{F_j\} \subset \mathcal{M}$ $\Gamma(V)$-converge to the functional $\tilde{F}$, let $x_0 \in X$ be fixed and let $\{w_j\} \subset V$ be the sequence of minimizers, i.e.

$$F_j(x_0 + w_j) \leqslant F_j(x_0 + v) \quad \forall v \in V, \ j = 1, 2, \ldots.$$

Analogous reasoning as in the proof of Theorem 1.5 gives that, if the sequence $\{w_j\}$ weakly converges to w_0, then

$$\begin{aligned} \tilde{F}(x_0 + w_0) &\leqslant \liminf_{j \to \infty} F_j(x_0 + w_j) \\ &\leqslant \inf\Big\{ \liminf_{j \to \infty} F_j(x_0 + v_j) \ \Big| \ \{v_j\} \subset V \Big\} \\ &\leqslant \tilde{F}(x_0 + v) \quad \forall v \in V, \end{aligned}$$

i.e. the element w_0 is a minimizer for $\tilde{F}(x_0 + \cdot)$ on V. Obviously, the sequence $\{w_j - w_0\}$ is the recovery sequence for $\tilde{F}(x_0 + w_0)$. That gives the convergence of "energies"

$$\min_{v \in V} F_j(x_0 + v) \to \tilde{F}(x_0 + w_0) \quad \text{as } j \to \infty.$$

3. Integral functionals of the calculus of variations

In this section, we briefly discuss two classes of integral functionals of the calculus of variations, to which the results of Section 2 can be applied.

Class 1. The vectorial case with different growth conditions.

Let $n \geqslant 2$ be integer, let $\Omega \subset \mathbf{R}^n$ be bounded Lipschitz domain and let $1 < q \leqslant p$ be given constants. Define the class $M(q, p, \nu, \mu, c)$ of Caratheodory integrands $f : \Omega \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $f = f(x, \zeta, \xi)$, such that

(i) $f(x, \cdot, \cdot)$ is convex on $\mathbf{R}^n \times \mathbf{R}^n$ for a.e. $x \in \Omega$;
(ii) $\nu|\zeta|^q + \nu|\xi|^p - c \leqslant f(x, \zeta, \xi) \leqslant \mu|\zeta|^q + \mu|\xi|^p + c$ a.e. $x \in \Omega$, $\forall (\zeta, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$.

The functionals $F : L_q(\Omega; \mathbf{R}^n) \times L_p(\Omega; \mathbf{R}^n) \to \mathbf{R}$,

$$F(a,b) := \int_\Omega f\big(x, a(x), b(x)\big)\,dx, \quad a \in L_q\big(\Omega; \mathbf{R}^n\big),\ b \in L_p\big(\Omega; \mathbf{R}^n\big), \tag{12}$$

with $f \in M(q, p, \nu, \mu, c)$ belong to the class $\mathcal{M}(\gamma_1, \gamma_2, c_1, c_2)$ where

$$\begin{aligned}
&X := L_q\big(\Omega; \mathbf{R}^n\big) \times L_p\big(\Omega; \mathbf{R}^n\big),\\
&\big\|(a,b)\big\| := \big(\|a\|_q^2 + \|b\|_p^2\big)^{1/2},\\
&\gamma_1 := 1/2\nu\big\|(a,b)\big\|^q, \quad \gamma_2 := 2\mu\big\|(a,b)\big\|^p,\\
&c_1 := \nu + c|\Omega|, \quad c_2 := \mu + c|\Omega|,
\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and $\|\cdot\|_q$, $\|\cdot\|_p$ denote the standard norms in Lebesgue spaces $L_q(\Omega; \mathbf{R}^n)$ and $L_p(\Omega; \mathbf{R}^n)$ respectively.

As the subspace V we choose

$$V := \left\{(u, v) \in X \;\middle|\; u = \nabla\varphi,\ \varphi \in W_q^1(\Omega),\ \int_\Omega \varphi(x)\,dx = 0,\ v = \nabla\psi,\ \psi \in W_p^1(\Omega),\ \int_\Omega \psi(x)\,dx = 0\right\}.$$

In this setting, since the space X is separable and uniformly convex, see, for instance, Day [7, p. 504], and V is a closed subspace of X, all assumptions of Theorems 1.4 and 1.5 are satisfied.

In some sense an inverse problem was discussed in [15], where for a quadratic functional in the space of divergence-free vectors it was shown that from an analogue of $\Gamma(\mathcal{N})$-convergence to a functional $\mathcal{F}$ it follows that the corresponding sequence of dual functionals converges to $\mathcal{F}^*$.

What concerns the representation of the $\Gamma(V)$-limit functional as an integral functional of the same type (12), the same reasoning as in Zhikov [17] can be applied with obvious modifications caused by the passage from the scalar case to the vectorial one.

Remark 3.1. One can replace the condition of the zero mean value in the definition of V by the condition that the functions φ and ψ take zero values on the boundary of Ω. Then the space $\mathcal{N}$ consists of div-free vector-functions and by using Theorems 1.4 and 1.5 one can obtain in a straightforward way a part of results from Ansini and Garroni [1], see, also Serrano [15], about the Γ-convergence of functionals defined on div-free fields.

Class 2. The case of additional integrands on the boundary of the reference domain.

Let $n \geqslant 2$ be integer, let $\Omega \subset \mathbf{R}^n$ be bounded Lipschitz domain with boundary $\partial\Omega$ and let $p \geqslant 2$, $0 < \nu \leqslant \mu$, c be fixed constants. Define the class $M(p, \nu, \mu, c)$ of pairs of Caratheodory functions (f, g), $f = f(x, \xi)$, $f : \Omega \times \mathbf{R}^n \to \mathbf{R}$, $g = g(x, u)$, $g : \partial\Omega \times \mathbf{R} \to \mathbf{R}$, such that

(iii) for a.e. $x \in \Omega$ $f(x, \cdot)$ is convex on $\mathbf{R}^n$ and for a.e. $x \in \partial\Omega$ $g(x, \cdot)$ is convex on $\mathbf{R}$;
(iv) $\nu|\xi|^2 - c \leqslant f(x, \xi) \leqslant \mu|\xi|^2 + c$ a.e. $x \in \Omega$, $\forall \xi \in \mathbf{R}^n$;
(v) $\nu|u|^p - c \leqslant g(x, u) \leqslant \mu|u|^p + c$ a.e. $x \in \partial\Omega$, $\forall u \in \mathbf{R}$.

With every pair (f, g) we associate a functional $F : X \to \mathbf{R}$, $X := L_2(\Omega; \mathbf{R}^n) \times L_p(\partial\Omega)$,

$$F(a,b) := \int_\Omega f\big(x, a(x)\big)\,dx + \int_{\partial\Omega} g\big(x, b(x)\big)\,dS, \quad (a,b) \in X. \tag{13}$$

We define the norm in X as

$$\big\|(a,b)\big\| := \big(\|a\|_2^2 + \|b\|_p^2\big)^{1/2}$$

and as the subspace V we choose

$$V := \big\{(v, u) \in X \;\big|\; v = \nabla\varphi,\ u = \gamma_0(\varphi),\ \varphi \in W_2^1(\Omega) \cap L_p(\partial\Omega)\big\},$$

where $\gamma_0 : W_2^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator.

Analogs of the space V appear in problems of heat transfer in conductive-radiative cases due the Stefan–Boltzmann law.

Since $p \geqslant 2$, then there exists a constant $c_* = c_*(\Omega, n, p)$ such that

$$\|\varphi\|_{W_2^1(\Omega)} \leqslant c_*\left[\left(\int_\Omega \big|\nabla\varphi(x)\big|^2\,dx\right)^{1/2} + \left(\int_{\partial\Omega} \big|\varphi(x)\big|^p\,dS\right)^{1/p}\right] \quad \forall \varphi \in W_2^1(\Omega) \cap L_p(\partial\Omega),$$

see, for instance, Nečas and Hlaváček [14, p. 69]. Therefore, the subspace V is closed and functionals F, defined by (13) with $(f,g) \in M(p,\nu,\mu,c)$ belong to the class $\mathcal{M}$ with

$$\gamma_1 := \nu \|(a,b)\|^2, \quad \gamma_2 := \mu\big(\|(a,b)\|\big)^p,$$
$$c_1 := \nu + c\big(|\Omega| + |\partial\Omega|\big), \quad c_2 := \mu + c\big(|\Omega| + |\partial\Omega|\big).$$

The subspace $\mathcal{N}$ for this case is

$$\mathcal{N} := \big\{(\eta,\sigma) \in L_2\big(\Omega;\mathbf{R}^n\big) \times L_{p/(p-1)}(\partial\Omega) \;\big|\; \operatorname{div}\eta = 0 \text{ in } \Omega,\ \sigma + \eta\cdot\bar{n} = 0 \text{ on } \partial\Omega\big\},$$

where $\bar{n}$ is the outward normal on $\partial\Omega$.

As far as we know, for analogs to $\mathcal{N}$ spaces the theory of Γ-convergence has not been developed completely yet. But the results of Section 2 give that for the case of convex functionals one can apply Theorem 1.4 for the subspace V and, after that, derive the representation for the $\Gamma(\mathcal{N})$-limit functional by using Theorem 1.5.

For convenience of the reader, we present below a brief sketch of proofs for the integral representation of $\Gamma(V)$-limit functionals in our case.

Let the sequence of functionals $\{F_j\}$,

$$F_j(a,b) := \int_\Omega f_j\big(x,a(x)\big)\,dx + \int_{\partial\Omega} g_j\big(x,b(x)\big)\,dS, \quad (a,b) \in X,\ j = 1,2,\ldots,$$

with integrands $(f_j,g_j) \in M(p,\nu,\mu,c)$, $j = 1,2,\ldots,$ $\Gamma(V)$-converges to a functional $\tilde{F}$.

Let a pair $(a,b) \in X$ with $|a(x)| \leqslant A$, $|b(x)| \leqslant A$ be fixed and let $\{(v_j,u_j) := (\nabla\varphi_j,\gamma_0(\varphi_j))\}$ be the recovery sequence for $\tilde{F}(a,b)$. Define truncated functions

$$\overline{\varphi}_j(x) = \begin{cases} \varphi_j(x), & \text{if } -A \leqslant \varphi_j(x) \leqslant A, \\ A, & \text{if } \varphi_j(x) > A, \\ -A, & \text{if } \varphi_j(x) < -A, \end{cases} \qquad j = 1,2,\ldots,$$

and the corresponding sequence $\{(\bar{v}_j,\bar{u}_j) := (\nabla\overline{\varphi}_j,\gamma_0(\overline{\varphi}_j))\}$. By construction, the sequence $(\bar{v}_j,\bar{u}_j)\}$ is eligible and exactly in the same way as in Zhikov [17, pp. 976–977] we have

$$\liminf_{j\to\infty} \int_\Omega f\big(x,a(x)+\bar{v}_j(x)\big)\,dx \leqslant \lim_{j\to\infty} \int_\Omega f\big(x,a(x)+v_j(x)\big)\,dx.$$

If the constant A is chosen such that $A > (2c\nu^{-1})^{1/p}$, then from assumptions (iii) and (v) it follows

$$g_j(x,A+\delta) > g_j(x,A), \quad g_j(x,-A-\delta) > g_j(x,-A) \quad \forall\delta > 0,\ j = 1,2,\ldots.$$

As a consequence, from here we have

$$\int_{\partial\Omega} g_j\big(x,b(x)+\bar{u}_j(x)\big)\,dS \leqslant \int_{\partial\Omega} g_j\big(x,b(x)+v_j(x)\big)\,dS, \quad j = 1,2,\ldots.$$

Thus $\{(\bar{v}_j,\bar{u}_j)\}$ is a recovery sequence for $\tilde{F}(a,b)$.

Due to the embedding theorems, the sequence $\{\bar{u}_j\}$ converges strongly in $L_2(\partial\Omega)$ to zero. By construction, all functions $\bar{u}_j$, $j = 1,2,\ldots,$ are uniformly bounded, hence, the sequence $\{\bar{u}_j\}$ converges to zero strongly in $L_p(\partial\Omega)$. Therefore, for every pair $(a,b) \in X \cap (L_\infty(\Omega;\mathbf{R}^n) \times L_\infty(\partial\Omega))$ and the corresponding recovery sequence $\{(v_j,u_j)\}$ there is

$$\lim_{j\to\infty} \int_{\partial\Omega} g_j\big(s,b(x)+u_j(x)\big)\,dS = \lim_{j\to\infty} \int_{\partial\Omega} g_j\big(x,b(x)\big)\,dS := G(b). \tag{14}$$

By virtue of properties (iii) and (v), all functions g_j, $j = 1,2,\ldots,$ are equi-coercive and uniformly locally Lipschitz

$$\big|g_j(x,b_1) - g_j(x,b_2)\big| \leqslant c_*(p,\mu,c)\big[1 + |b_1|^{p-1} + |b_2|^{p-1}\big]|b_1 - b_2| \quad \forall b_1,b_2 \in \mathbf{R}.$$

From here and the separability of X it follows immediately that the relationship (14) holds for all $(a,b) \in X$. By construction, the functional G is well defined on $L_p(\partial\Omega)$, it is convex and it satisfies growth conditions

$$\nu\|b\|_p^p - c|\partial\Omega| \leqslant G(b) \leqslant \mu\|b\|_p^p + c|\partial\Omega|, \quad b \in L^p(\partial\Omega).$$

Let $\mathcal{L} = \{E\}$ be algebra of Lebesgue measurable subsets of $\partial\Omega$ and let the functional $\hat{G} :\to \mathcal{L} \times L_p(\partial\Omega) \to \mathbf{R}$ be defined as

$$\hat{G}(E,b) := \lim_{j\to\infty} \int_E g_j\big(x,b(x)\big)\,dS.$$

By construction, the mapping $\hat{G}$ is well defined, it is local, for every fixed $b \in L_p(\partial\Omega)$ it is a measure, and the mapping $\hat{G}(\partial\Omega, \cdot) = G(\cdot) : L_p(\partial\Omega) \to \mathbf{R}$ is continuous. These properties are sufficient for that the mapping $\hat{G}$ has the representation

$$\hat{G}(E, b) = \int_E \tilde{g}\big(x, b(x)\big)\, dS$$

with some Caratheodory function $\tilde{g}$, see Buttazzo and Dal Maso [4, pp. 493–494] or Fonseca and Leoni [10, pp. 464–465]. From here and established properties of the functional G it follows immediately that the function $\tilde{g}$ belongs to $M(p, \nu, \mu, c)$.

On the other hand, the De Giorgi method for matching boundary conditions, see, for instance, Braides [3, p. 129] or Dal Maso [6, pp. 208–210], gives that the $\Gamma(V)$-limit problem for the integral over Ω can be separated from the limit problem for the integral over $\partial\Omega$. Therefore, if we split functionals $F : X \to \mathbf{R}$ as

$$F(a, b) := F^1(a) + F^2(b), \quad F^1(a) := \int_\Omega f\big(x, a(x)\big)\, dx, \quad F^2(b) := \int_{\partial\Omega} g\big(x, b(x)\big)\, dS,$$

then $\tilde{F}$ has the representation

$$\tilde{F}(a, b) = \tilde{F}^1(a) + \tilde{F}^2(b),$$

where

$$\tilde{F}^2(b) = \int_{\partial\Omega} \tilde{g}\big(x, b(x)\big)\, dS, \quad b \in L_p(\partial\Omega),$$

but $\tilde{F}^1$ is the $\Gamma(V^1)$-limit for the sequence of functionals

$$F^1_j(a) := \int_\Omega f_j\big(x, a(x)\big)\, dx, \quad a \in L_2\big(\Omega, \mathbf{R}^n\big),\ j = 1, 2, \ldots,$$

with

$$V^1 := \left\{ v \in L_2\big(\Omega; \mathbf{R}^n\big) \,\middle|\, v = \nabla\varphi,\ \varphi \in W^1_2(\Omega),\ \int_\Omega \varphi(x)\, dx = 0 \right\}.$$

In its turn, the integral representation for $\tilde{F}^1$,

$$\tilde{F}^1(a) = \int_\Omega \tilde{f}\big(x, a(x)\big)\, dx, \quad a \in L_2\big(\Omega; \mathbf{R}^n\big),$$

with some Caratheodory function $\tilde{f}$ that satisfies (iii) and (iv) was shown in Zhikov [17, p. 972].

Thus the $\Gamma(V)$-limit functional $\tilde{F}$ for our sequence belongs to $\mathcal{M}(p, \nu, \mu, c)$.

Example 3.2. Let $\Omega \subset \mathbf{R}^n$, $n \geqslant 3$, be bounded Lipschitz domain and let $p \geqslant 2n/(n-2)$. Due to the embedding theorems there exists a sequence $\{v_k\} \subset H^1(\Omega) \cap L_p(\Omega)$ such that

$$\|v_k\|_{L_p(\Omega)} = 1, \quad k = 1, 2, \ldots;$$

$$v_k \to 0 \quad \text{strongly in } H^1(\Omega) \quad \text{and} \quad \text{weakly in } L_p(\Omega) \quad \text{as } k \to \infty.$$

Define functionals I_k,

$$I_k(a, b) := \int_\Omega \left[\sum_{i=1}^n (a_i - v_{kx_i})^2 + |b - v_k|^p \right] dx, \quad (a, b) \in L_2\big(\Omega; \mathbf{R}^n\big) \times L_p(\Omega),\ k = 1, 2, \ldots,$$

and let the subspace V be defined as

$$V := \big\{ (w, w_0) \in L_2\big(\Omega; \mathbf{R}^n\big) \times L_p(\Omega) \,\big|\, w_i = w_{0x_i},\ i = 1, \ldots, n;\ w_0 \in H^1(\Omega) \cap L_p(\Omega) \big\}.$$

By construction, to the sequence $\{I_k\}$ apply Theorems 1.3 and 1.4, and the sequence $\Gamma(V)$-converges to the functional I_0,

$$I_0(a, b) := \int_\Omega \left[\sum_{i=1}^n a_i^2 + |b|^p \right] dx.$$

In its turn, it is easy to check that $\{I_k\}$ does not possess Mosco convergent subsequences on $L_p(\Omega)$.

References

[1] N. Ansini, A. Garroni, Γ-convergence of functionals on div-free fields, ESAIM Control Optim. Calc. Var. 13 (2007) 809–828.

[2] G. Beer, On the Young–Fenchel transform for convex functions, Proc. Amer. Math. Soc. 104 (1988) 1115–1123.

[3] A. Braides, A handbook of Γ-convergence, in: M. Chipot, P. Quittner (Eds.), Handbook of Differential Equations Stationary, in: Partial Differential Equations, vol. 3, Elsevier, The Netherlands, 2006, pp. 101–214.

[4] G. Buttazzo, G. Dal Maso, On Nemitskii operators and integral representation of integral functionals, Rend. Mat. 3 (1983) 491–509.

[5] C. Carstensen, A posteriori error estimate for the mixed finite element methods, Math. Comp. 66 (1997) 465–476.

[6] G. Dal Maso, An Introduction to Γ-Convergence, Birkhäuser, Boston, 1993.

[7] M.M. Day, Some more uniformly convex spaces, Bull. Amer. Math. Soc. 47 (1941) 504–507.

[8] N. Dunford, J.T. Schwartz, Linear Operators. Part I. General Theory, Interscience Publishers, New York, 1958.

[9] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, North-Holland, Amsterdam, 1976.

[10] I. Fonseca, G. Leoni, Modern Methods in the Calculus of Variations: L^p Spaces, Springer, New York, 2007.

[11] W. Han, A-posteriori error analysis for linearization of nonlinear elliptic problems and their discretizations, Math. Methods Appl. Sci. 17 (1994) 487–535.

[12] D.P. Milman, On some criteria for regularity of spaces of the type (B), Dokl. Akad. Nauk SSSR 20 (1938) 243–246.

[13] U. Mosco, On the continuity of the Young–Fenchel transform, J. Math. Anal. Appl. 35 (1971) 518–535.

[14] I. Nečas, I. Hlaváček, Úvod do matematické teorie pružných a pružné plastických těles, Nakladatelství technické literatury, Praha, 1983.

[15] H. Serrano, On Γ-convergence in divergence-free fields through Young measures, J. Math. Anal. Appl. 359 (2009) 311–321.

[16] S. Repin, A posteriori error estimation for variational problems with uniformly convex functionals, Math. Comp. 69 (1999) 481–500.

[17] V.V. Zhikov, Questions of convergence, duality and averaging for functionals of calculus of variations, Izv. Akad. Nauk SSSR Ser. Math. 47 (1983) 961–998 (in Russian).